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INFINITE CLUSTERS AND CRITICAL VALUES IN TWO-DIMENSIONAL CIRCLE PERCOLATION

BY

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ABSTRACT

We consider a dependent percolation model on $\mathbf{Z}^2$ that does not have the 'finite energy' property. It is shown that the number of infinite clusters equals zero, one or infinity. Furthermore, we investigate a dynamical system which is associated with the calculation of the critical value in this model. It is shown that for almost all choices of the parameters in the model, this critical value can be calculated in a finite number of iterations.

1. Introduction

In [M], we introduced a dependent parametric percolation model on $\mathbf{Z}^d$, called circle percolation. Following tradition in percolation theory, our first results concerned the calculation of critical values and percolation functions. It was shown that in this model, these objects can in principal be calculated explicitly. The question that remained considered the *finiteness* of the calculations. It was conjectured that for almost all choices of the parameters in the model, the critical value can be obtained in a finite number of iterations of a certain algorithm.

In this paper, we consider the model on $\mathbf{Z}^2$ only, and we will prove the conjecture in this case. Furthermore, we will use the special properties of the plane to study infinite clusters in this model. Only recently, R. M. Burton and M. Keane ([BKa]) proved that a stationary percolation model on $\mathbf{Z}^d$, having the so-called 'finite energy property', can have at most one infinite open cluster with probability one. However, our model, which will be defined in the

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following section, does not have this property and it will turn out that the number of infinite open clusters can be equal to zero, one or infinity.

In [BKb], topological properties of infinite clusters in two-dimensional stationary percolation models were studied. It was shown that, basically, an infinite cluster is either a topological strip, half-plane or plane. We prove in Section 4 that in our model, which is stationary, infinite clusters are either *contained* in strips, or basically the whole plane, which is in nice agreement with the results in [BKb]. Clusters like half-planes do not occur in our model.

The paper is organized as follows. In Section 2, we define our model and we summarize those results from [M] which we need here. Sections 3 and 4 are devoted to an analysis of the structure and number of infinite clusters. The statements mentioned above will be proved there. The final section is devoted to a proof of the conjecture mentioned in the first paragraph.

2. The model

We will discuss a percolation model on $\mathbf{Z}^2$ and we start by fixing notation and terminology in this space. Elements of $\mathbf{Z}^2$, called vertices, will usually be denoted by z or z', the unit vectors by e_1 and e_2 respectively, and the null vector by $\mathbf{0}$. A *path* in $\mathbf{Z}^2$ is a (possibly infinite) sequence $\pi = (\pi_0, \pi_1, \ldots)$ such that $\pi_i \in \mathbf{Z}^2$ and $d(\pi_i, \pi_{i+1}) = 1$, for all $i \geqq 0$, where d denotes Euclidean distance. A *circuit* is a path $(\pi_0, \ldots, \pi_n)$ such that $\pi_0 = \pi_n$ and $\pi_i \neq \pi_j$ for all $(i, j) \neq (0, n)$. A path is called *self-avoiding* if it contains no circuits. The set of all infinite self-avoiding paths is denoted by Π.

In percolation theory, one studies random configurations of $\mathbf{Z}^2$ in which each vertex can be in two possible states: it is either *open* or *closed*. An *open path* is a path whose elements are all open. Two open vertices z and z' are said to be *connected* if there exists an open path $(\pi_0, \ldots, \pi_n)$ such that $\pi_0 = z$ and $\pi_n = z'$. An *open cluster* is a maximal set of connected open vertices. Analogous definitions can be made with 'closed' instead of 'open'. Typical questions in percolation theory are the following: (i) Does there exist an infinite open/closed cluster? (ii) Is $\mathbf{0}$ (or any other particular vertex) contained in an infinite cluster? (iii) How many infinite clusters arise?

The answers to these questions of course depend on the random mechanism which generates the configurations in $\mathbf{Z}^2$. In this paper, we study the following model which we have called 'circle percolation' for obvious reasons:

Let $\Omega = \mathbf{R}/\mathbf{Z}$, $\mathscr{A}$ be the ordinary Borel σ-field and μ be Lebesgue measure on Ω. $(\Omega, \mathscr{A}, \mu)$ is our probability space. Let $0 \leqq \alpha \leqq \beta \leqq \frac{1}{2}$ be two parameters, subject to the condition that $k\alpha + l\beta = m$ for some $k, l, m \in \mathbf{Z}$ implies $k = l = m = 0$. This condition is imposed only to avoid messy statements and is not at all essential. It is also the most interesting case because the associated configurations are not periodic in any direction.

Now define, for $0 \leqq p < 1$, the random variable $I_p : \Omega \to \{\text{open, closed}\}^{\mathbf{Z}^2}$ as follows:

We declare $z = (z_1, z_2) \in \mathbf{Z}^2$ to be open in $I_p(\omega)$ if

$$z(\omega) := \omega + z_1\alpha + z_2\beta \ (\text{mod } 1) \leqq p,$$

otherwise z is closed. It is easily seen that this is a stationary model. It is one of the few non-trivial percolation models in which the critical value — the smallest p for which there is a positive probability for the origin to be in an infinite open cluster — can be calculated; see [M] and Theorem 2.2 below. Further motivation follows from the following observation. Consider the following analogous model. Let Ω' be the space $[0, 1]^{\mathbf{Z}^2}$ and μ' product Lebesgue measure on Ω'. Define shifts $T_1, T_2 : \Omega' \to \Omega'$ by $(T_1\omega)_{i,j} = \omega_{i+1,j}$ and $(T_2\omega)_{i,j} = \omega_{i,j+1}$. Let $A_p = \{\omega \in \Omega' \mid \omega_{00} \leqq p\}$. Let $z = (z_1, z_2) \in \mathbf{Z}^2$ be open iff $(T_1)^{z_1}(T_2)^{z_2}(\omega) \in A_p$. Then we have a model for ordinary independent site percolation on $\mathbf{Z}^2$. The analogy between the two models is clear, and this was the original motivation to study circle percolation.

We remark that in this type of model, all independent percolation models with parameter $p \in [0, 1]$ are coupled together. In the literature, it is used for example in Hammersley ([H]) for simulation purposes and in van den Berg and Keane ([BK]), where results on the continuity of the percolation function are proved.

The first and second questions above were treated in [M], and the first part of this paper is concerned with the third. To make the paper self-contained, we recall some definitions and results from [M]. Let

$$\mathscr{P}_p = \mathscr{P}_p(\alpha, \beta) = \{\omega \in \Omega \mid C_{p,\omega} \text{ is infinite}\},$$

where $C_{p,\omega}$ denotes the open cluster in $I_p(\omega)$ which contains $\mathbf{0}$. ($C_{p,\omega}$ is empty if $\mathbf{0}$ is closed.) $\mathscr{P}_p$ is called the *percolation set* of $[0, p]$. Obviously, $\omega \in \mathscr{P}_p$ iff there exists an infinite self-avoiding open path of the form $\pi = (\mathbf{0}, \pi_1, \pi_2, \ldots)$ in $I_p(\omega)$. In such a case, we say that ω percolates in $[0, p]$ along π. The measurability of $\mathscr{P}_p$ was proved in [M] and of particular interest is the *critical value*, which is defined as

$$p_c = p_c(\alpha, \beta) = \inf\{p \mid \mu(\mathscr{P}_p) > 0\}.$$

The next lemma can be useful.

LEMMA 2.1 (cf. [M], Lemma 4.1).

$$p_c = \inf\{ p \mid \mathscr{P}_p \neq \emptyset \}.$$

PROOF. If $\omega \in \mathscr{P}_p$ for some p, it is clear that $[\omega, \omega+\varepsilon] \subset \mathscr{P}_{p+\varepsilon}$, for all $\varepsilon > 0$, because $C_{p+\varepsilon,\omega'} \supset C_{p,\omega}$ for all $\omega' \in [\omega, \omega+\varepsilon]$. So if $\mathscr{P}_p \neq \emptyset$, then $\mu(\mathscr{P}_{p+\varepsilon}) > 0$ for all $\varepsilon > 0$ and we conclude that $p_c = \inf\{ p \mid \mathscr{P}_p \neq \emptyset \}$. ■

The following dynamical system is associated with the calculation of p_c.

Let $\mathcal{O} = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid 0 \leqq x_1 \leqq x_2 \leqq x_3\}$ and define $M : \mathcal{O} \to \mathcal{O}$ as follows: $M(x_1, x_2, x_3) = (y_1, y_2, y_3)$, where y_1, y_2 and y_3 are the numbers x_1, $x_2 - x_1$ and $x_3 - x_1$ arranged in increasing order. We now have the following result:

THEOREM 2.2. *Let* $(\alpha_0, \beta_0, \gamma_0) = (\alpha, \beta, 1-\beta)$ *and let* $(\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}) = M(\alpha_i, \beta_i, \gamma_i)$, *for all* $i \geqq 0$. *If*

$$(*) \qquad \alpha_{i_0} + \beta_{i_0} \leqq \gamma_{i_0}, \qquad \text{for some } i_0 \in \mathbf{N},$$

then

$$p_c = \sum_{i=0}^{i_0} \alpha_i + \beta_{i_0} < \frac{1+\alpha}{2},$$

otherwise

$$p_c = (1+\alpha)/2.$$

We always have $p_c \leqq 1 - \alpha$.

PROOF. See the proofs of Theorem 4.8 and Lemma 4.3 in [M]. ■

The first question that springs to mind reading this theorem is the following: Is it possible to decide whether or not (*) holds? Section 5 is devoted to this problem. In particular, we will show that for Lebesgue almost all choices of $(\alpha, \beta) \in \mathbf{R}^2$, the condition holds.

Now we assume that (*) holds, and let $y = \sum_{i=0}^{i_0-1} \alpha_i = p_c - (\alpha_{i_0} + \beta_{i_0})$. We showed in [M] that $\mathscr{P}_p$ is a finite union of closed intervals which can be determined precisely. In particular, $\mu(\mathscr{P}_{p_c}) > 0$ so we have a *discontinuous* phase-transition. Now let $\omega \in \mathscr{P}_{p_c}$ and let $z_0 \in C_{p_c,\omega}$ such that $z_0(\omega) \geqq y$. Define, for all $\eta \in [y, p_c]$, $\mathscr{R}_\eta$ as the set of points η' for which there exists a finite chain $(d_0 = \eta, d_1, \ldots, d_N = \eta')$ such that $d_i \neq d_j$ if $i \neq j$, $d_i \in [y, p_c]$ for all i, and $d_{i+1} - d_i \in \{\pm \alpha_{i_0}, \pm \beta_{i_0}\}$. A little thought yields the conclusion that in such a chain either $d_{i+1} - d_i \in \{\alpha_{i_0}, -\beta_{i_0}\}$ for all i or $d_{i+1} - d_i \in \{-\alpha_{i_0}, \beta_{i_0}\}$ for all i. We now have the following result:

LEMMA 2.3. *Let $z_0(\omega)$ be as above. Then*

$$\mathcal{R}_{z_0(\omega)} = \{z(\omega) \mid z \in C_{p_c,\omega}, z(\omega) \geqq y\}.$$

PROOF. This follows from the proof of Theorem 4.8 in [M]. ■

3. A second critical value

One of the purposes of this paper is to determine the number of infinite clusters in two-dimensional circle percolation. For this we should be able to decide whether or not two given open vertices are in different open clusters. As an example, let

$$C^* = \bigcup_{n \in \mathbf{Z}} \{(0, 2n) \cup (1, 2n+1)\},$$

and consider the configuration in which all sites in C^* are closed and all sites in $\mathbf{Z}^2 \setminus C^*$ are open; see Fig. 1. It is clear that in this situation there are two infinite open clusters, one on each side of C^*. They are separated by C^*, which is *not* an infinite closed cluster itself. So to get two infinite open clusters it is not necessary to have an infinite closed cluster. We now have to make some new definitions. A *star-path* is defined as a normal path but now we only require that $d(\pi_i, \pi_{i+1}) \leqq \sqrt{2}$ for all $i \geqq 0$. A *star-circuit* is a finite star-path $\pi^* =$

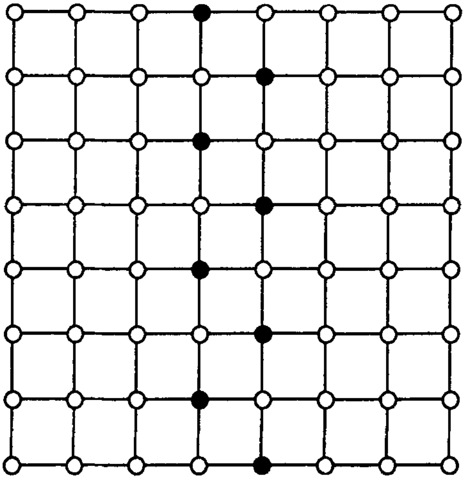

Fig. 1. Elements of C^* are black, the others white.

$(\pi_0^*, \ldots, \pi_n^*)$ such that $\pi_0^* = \pi_n^*$, $\pi_i^* \neq \pi_j^*$ for all $(i, j) \neq (0, n)$. Two open vertices are said to be *star-connected* if there exists an open finite star-path joining them. An *open star-cluster* is a maximal set of star-connected open vertices. Closed star-paths and closed star-clusters are defined analogously. The set C^* above, for example, is a closed infinite star-cluster. The notion of star-connections appears in the literature in [SE] and [R] for example.

The following remark is obvious in the light of the example above: Let S be a strip and consider a configuration in which all vertices not in S are open and in which S contains a closed star-cluster which is unbounded in both directions of S. Then any two vertices on different sides of S are not connected. If S does not contain such a star-cluster then any such two vertices are connected.

The first step is to find out when infinite star-clusters arise in $I_p(\omega)$ for $0 \leqq p < 1$ and $\omega \in \Omega$. For this we define

DEFINITION 3.1. For $0 \leqq p < 1$,

$$\mathscr{P}_p^* = \{\omega \in \Omega \mid C_{p,\omega}^* \text{ is infinite}\},$$

where $C_{p,\omega}^*$ denotes the open star-cluster in $I_p(\omega)$ which contains $\mathbf{0}$.

It is possible to prove the measurability of $\mathscr{P}_p^*$ directly, but this also follows from Theorem 3.3 below. Before we state and prove this theorem, we first prove the following simple lemma, needed in the proof of Theorem 3.3.

LEMMA 3.2. *Let $\omega \in \Omega$ and $p \geqq \alpha$. Let π be a path such that $\pi_n(\omega) \leqq p$ for all n. If $\pi_i(\omega) \in (p - \alpha, p]$ for some $i \geqq 1$ then*

$$\pi_{i \pm 1}(\omega) \in [0, p - \alpha].$$

PROOF. Let $l_i = \pi_{i+1} - \pi_i$ for $i \geqq 0$. Suppose $\pi_{i+1}(\omega) \in (p - \alpha, p]$. We will show that this leads to a contradiction. For this, we consider the four possible cases. If $l_i = e_1$, we obtain by translation $p + \alpha > 1 + p - \alpha$, which means that $2\alpha > 1$. This is impossible by our choice of α. If $l_i = -e_1$, we obtain $p - \alpha - \alpha < -1 + p$ which again leads to $2\alpha > 1$. The case in which $l_i = \pm e_2$ is treated analogously. ∎

The following theorem tells us how to determine $\mathscr{P}_p^*$ once $\mathscr{P}_p$ is known. The measurability of $\mathscr{P}_p^*$ follows immediately.

THEOREM 3.3. (i) *If $p < 1 - \alpha$, then $\mathscr{P}_p^* = \mathscr{P}_{p+\alpha} \cap [0, p]$.*
(ii) *If $p \geqq 1 - \alpha$, then $\mathscr{P}_p^* = [0, p]$.*

PROOF. (i) Suppose $\omega \in \mathscr{P}_p^*$ and ω star-percolates in $[0, p]$ along π^*. Then

π_i^* is open in $I_p(\omega)$, for all i, so $\pi_i^* + e_1$ is open in $I_{p+\alpha}(\omega)$. If $d(\pi_i^*, \pi_{i+1}^*) = \sqrt{2}$ for some i, it is clear that π_i^* and π_{i+1}^* are connected in $I_{p+\alpha}(\omega)$ via either $\pi_i^* + e_1$ or $\pi_{i+1}^* + e_1$. By induction, we conclude that for all $i, j \in \mathbf{N}$, π_i^* is connected with π_j^* in $I_{p+\alpha}(\omega)$. This implies that $C_{p+\alpha,\omega}$ is infinite and we conclude that $\omega \in \mathscr{P}_{p+\alpha}$.

For the reverse inclusion, suppose $\omega \in \mathscr{P}_{p+\alpha} \cap [0, p]$ and ω percolates in $[0, p+\alpha]$ along π. Define a subsequence $(n_0, n_1, \ldots)$ of $(0, 1, 2, \ldots)$ such that $n_0 = 0$, $n_{i+1} = \min\{n > n_i \mid \pi_n(\omega) \leqq p\}$, for all $i \geqq 0$. We claim that in $I_p(\omega)$, π_{n_i} is star-connected with $\pi_{n_{i+1}}$, for all i. Once we have shown this, we conclude that π_{n_i} is star-connected with π_{n_j} in $I_p(\omega)$, for all i and j. This implies that $C_{p,\omega}^*$ is infinite and thus $\omega \in \mathscr{P}_p^*$. So it remains to prove the claim. It follows from Lemma 3.2 that $d(\pi_{n_i}, \pi_{n_{i+1}}) \leqq 2$ for all i. If $d(\pi_{n_i}, \pi_{n_{i+1}}) \leqq \sqrt{2}$ then π_{n_i} and $\pi_{n_{i+1}}$ are trivially star-connected in $I_p(\omega)$. If $d(\pi_{n_i}, \pi_{n_{i+1}}) = 2$, we have the following possibilities.

(I) $\pi_{n_{i+1}} - \pi_{n_i} = 2e_1$,
(II) $\pi_{n_{i+1}} - \pi_{n_i} = -2e_1$,
(III) $\pi_{n_{i+1}} - \pi_{n_i} = 2e_2$,
(IV) $\pi_{n_{i+1}} - \pi_{n_i} = -2e_2$,

Case (I). Let j be defined by $n_i < j < n_{i+1}$. Now $\pi_j(\omega) \geqq 1 - \alpha \geqq 1 - \beta$ and it follows that $(\pi_j + e_2)(\omega) \in [\pi_{n_{i+1}}(\omega), \pi_{n_i}(\omega)]$, using the fact that $0 \leqq \alpha \leqq \beta \leqq \frac{1}{2}$. This means that $\pi_j + e_2$ is open in $I_p(\omega)$ and we conclude that π_{n_i} and $\pi_{n_{i+1}}$ are star-connected in $I_p(\omega)$ via $\pi_j + e_2$; see Fig. 2.

Case (II). An analogous argument again shows that $\pi_j + e_2$ is open in $I_p(\omega)$ and the claim follows.

Case (III), (IV). An analogous argument shows that $\pi_j - e_1$ is open in $I_p(\omega)$ and, again, π_{n_i} and $\pi_{n_{i+1}}$ are star-connected via $\pi_j - e_1$; see Fig. 2. This proves the claim.

(ii) Consider any $\omega \in [0, p]$ and let

$$\pi = (0, e_1, e_1 + e_2, 2e_1 + e_2, 2e_1 + 2e_2, \ldots).$$

It is easy to see that if $\pi_i(\omega) > p$ for some $i \geqq 1$ then $\pi_{i \pm 1}(\omega) \leqq p$. So the sequence $\pi^* = (0, \pi_{k_1}, \pi_{k_2}, \ldots)$, where $k_1 = \min\{n > 0 \mid \pi_n(\omega) \leqq p\}$ and $k_{i+1} = \min\{n > k_i \mid \pi_n(\omega) \leqq p\}$ for all $i \geqq 0$, is an infinite self-avoiding open star-path in $I_p(\omega)$ and we conclude that $\omega \in \mathscr{P}_p^*$. As the reverse inclusion is trivial, this finishes the proof. ∎

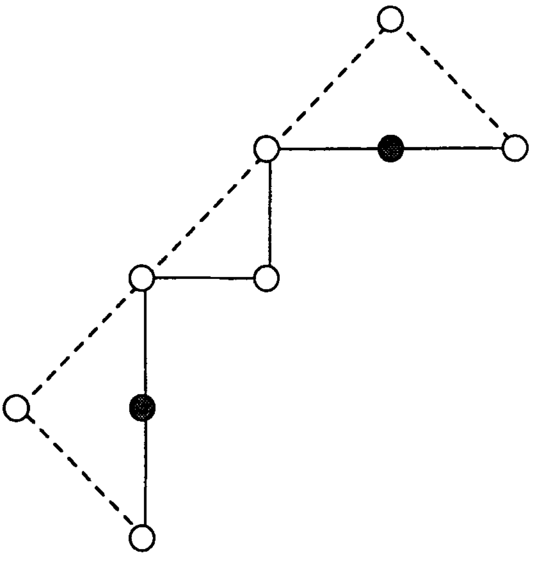

Fig. 2. Construction of an open star-path (dotted line) from an open path (solid line). The grey vertices are open in $I_{p+\alpha}(\omega)$ but closed in $I_p(\omega)$.

Now we can define a second critical value.

DEFINITION 3.4. $p_c^* = p_c^*(\alpha, \beta) = \inf\{ p \mid \mu(\mathscr{P}_p^*) > 0\}$.

Using Theorem 3.3, it is easy to calculate p_c^*:

COROLLARY 3.5. $p_c^*(\alpha, \beta) = p_c(\alpha, \beta) - \alpha$.

PROOF. As in Lemma 2.1, we have $p_c^* = \inf\{ p \mid \mathscr{P}_p^* \neq \varnothing \}$. This, together with Theorem 3.3 and the fact that $p_c \leqq 1 - \alpha$, proves the corollary. ■

In Section 2 we saw that $p_c \leqq (1 + \alpha)/2$. It now follows, using Corollary 3.5, that $p_c \leqq 1 - p_c^*$. Equality only occurs if $p_c = (1 + \alpha)/2$ and this is the case iff (*) in Theorem 2.2 does not hold.

The critical value p_c^* appears in the literature in the context of ordinary independent site percolation (see [SE], [R]). The relation between p_c^* and p_c in that model is given by $p_c + p_c^* = 1$ (see [R]). In our model strict inequality is valid in most cases. This suggests that it is possible to have infinite open clusters and infinite closed star-clusters simultaneously, if p is between the two critical values.

4. The number of infinite clusters

In this section, we will determine the number of infinite open clusters in $I_p(\omega)$ as a function of p and ω. By symmetry, the number of infinite closed clusters then follows immediately.

Only recently, R. M. Burton and M. Keane (see [BKb]) proved that in a stationary model that has the 'finite energy property' (see e.g. [NS]), the number of infinite open clusters equals zero or one with probability one. It is easy to show directly that our model does not have this property but this also follows from the following theorem.

Let $A_p(\omega)$ denote the number of infinite open clusters in $I_p(\omega)$. Then we have

THEOREM 4.1. (i)

$$A_p(\omega) = \begin{cases} 0, & p < p_c, \\ 1, & p > 1 - p_c^*, \end{cases}$$

for all $\omega \in \Omega$.

(ii) *If* $(*)$ *in Theorem* 2.2 *holds, then*

$$A_p(\omega) = \infty, \qquad p_c \leqq p \leqq 1 - p_c^*,$$

for all $\omega \in \Omega$.

In particular, $A_p(\omega)$ *does not depend on* ω.

If $(*)$ does not hold and $p = p_c = 1 - p_c^*$ we do not know the number of infinite clusters, but we expect it to be infinity.

To prove the theorem, we need the following lemma which we will state as being 'obvious'. A proof can be found in [K], Section 2.4, where the lemma follows as a special case of a more general result. It is possible to give an easier proof in this situation which proceeds by induction, but we leave this to the reader. Let, for any finite (star)-cluster C, the unique infinite component of $\mathbf{Z}^2 \setminus C$ be denoted by ext(C).

LEMMA 4.2. *Consider a configuration of the square lattice. Then any finite open cluster* C *is surrounded by a closed star-circuit, which can be written as* $\{z \in \text{ext}(C) \mid d(z, C) = 1\}$. *Furthermore, any finite open star-cluster* C^* *is surrounded by a closed circuit, which can be written as*

$$\{z \in \text{ext}(C^*) \mid d(z, C^*) \leqq \sqrt{2}\}.$$

The same is true with 'open' and 'closed' interchanged.

We start with the easy proof of Theorem 4.1(i).

PROOF OF THEOREM 4.1(i). The first assertion is trivial as for $p < p_c$ there cannot be an infinite open cluster because $\mathscr{P}_p = \emptyset$ for $p < p_c$, see Lemma 2.1. Now let $\omega \in \Omega$ and $p > 1 - p_c^*$. Consider two arbitrary open vertices z and z' which are in an infinite open cluster. Such vertices exist because $p > p_c$ and $\mu(\mathscr{P}_p) > 0$ for all $p \geqq p_c$. Now consider any path joining these two vertices. Denote this path by $\pi = (\pi_0, \ldots, \pi_n)$ where $\pi_0 = z$ and $\pi_n = z'$. Consider the first element of this path, π_{i_0} say, which is closed and let $i_0' := \min\{n > i_0 \mid \pi_n \text{ is open}\}$. Now π_{i_0} is in a finite closed star-cluster and is therefore surrounded by an open circuit as in Lemma 4.2. It is easy to see that π_{i_0-1} and $\pi_{i_0'}$ are elements of this circuit and hence are connected by an open path. It is now clear how to construct an open path joining z and z' which implies that z and z' are in the same cluster. Because z and z' were arbitrary, this means that the infinite open cluster is unique and Theorem 4.1(i) follows. ∎

In the proof of Theorem 4.1(ii), we need the fact that finite clusters cannot be too large in $I_p(\omega)$ for $\omega \in \Omega$ and $p < p_c$. The following lemma deals with this property.

LEMMA 4.3. *Let $\omega \in \Omega$ and $p < p_c$. Then there exists a constant $K = K(p)$ depending on p alone, such that for any two connected open vertices z and z', we have $d(z, z') < K$.*

PROOF. Suppose not. Then, for any $i \in \mathbf{N}$, there exist $\omega_i \in \Omega$ and $\pi^i \in \Pi$ such that $\pi_n^i(\omega_i) \leqq p$ for all $i \leqq n_i$, where $\lim_{i\to\infty} n_i = \infty$. Let ω be a limit point of $\{\omega_i\}$: $\lim_{i\to\infty} \omega_{k_i} = \omega$, where $(k_1, k_2, \ldots)$ is some subsequence of $(1, 2, \ldots)$. Now $\pi_1^{k_i}$ can take only 4 values so there is at least one $\pi_1 \in \mathbf{Z}^2$ such that $\pi_1^{k_i} = \pi_1$ i.o. Select a further subsequence $(k_1', k_1', \ldots)$ of $(k_1, k_2, \ldots)$ such that $\pi_1^{k_i'} = \pi_1$, for all i. Now there again is at least one $\pi_2 \in \mathbf{Z}^2$ such that $\pi_2^{k_i'} = \pi_2$ i.o. Proceeding in this way, we obtain a sequence $\pi = (\pi_1, \pi_2, \ldots)$. It is easy to see that $\pi_i(\omega) \leqq p$ for all i and that $\pi \in \Pi$, so $\omega \in \mathscr{P}_p$. But $p < p_c$ so this is a contradiction, using Lemma 2.1. ∎

The most important step towards a proof of Theorem 4.1(ii) is the following lemma, which implies that we can control the shape of infinite clusters.

LEMMA 4.4. *Let $\omega \in \Omega$ and consider $I_{p_c}(\omega)$. Suppose $(*)$ in Theorem 2.2* *holds. Then each infinite open cluster is contained in a strip, and is unbounded in both directions of the strip.*

PROOF. Let y be as in Section 2. Choose $z_0 \in C_{p_c,\omega}$ such that $z_0(\omega) \geqq y$. According to Lemma 2.3 and the remark before this lemma, we can write $\{z \in C_{p_c,\omega} \mid z(\omega) \geqq y\} = R \cup R'$, where all elements of R correspond with chains using $-\alpha_{i_0}$ and β_{i_0}, and elements of R' correspond with chains using α_{i_0} and $-\beta_{i_0}$. Now we can order the elements in R and R' according to the length of the chain from z_0 to those elements: $R = (z_1, z_2, \ldots)$ and $R' = (z'_1, z'_2, \ldots)$. It is not difficult to see that there exist two vectors v_α and v_β such that $z_{i+1} - z_i \in \{-v_\alpha, v_\beta\}$ and $z'_{i+1} - z'_i \in \{v_\alpha, -v_\beta\}$ for all i. Consider the line

$$l = \{z \in \mathbf{Z}^2 \mid z = z_0 + \lambda(\beta_{i_0} v_\alpha + \alpha_{i_0} v_\beta), \lambda \in \mathbf{R}\}.$$

Now z_i lies above l iff $z_i(\omega) > z_0(\omega)$, and below l iff $z_i(\omega) < z_0(\omega)$. Analogous statements can be made for z'_i. Furthermore, the maximum number of subsequent elements z_i which are all below or all above l is easily seen to be bounded and the same is true for elements z'_i. We conclude that $\{z \in C_{p_c,\omega} \mid z(\omega) \geqq y\}$ is contained in a strip S say, containing l. In addition, z_i and z'_i tend to infinity in opposite directions of S if i tends to infinity.

Now let z be any vertex in $C_{p_c,\omega}$, let $\omega' := z(\omega)$ and π be such that ω' percolates in $[0, p_c]$ along π. According to Lemma 4.3, there is a number $k < K = K(y)$ such that $\pi_k(\omega') \geqq y$, where K does not depend on z. This is to say that there exists a $z^* \in \{z \in C_{p_c,\omega} \mid z(\omega) \geqq y\}$ such that $d(z, z^*) \leqq K$. This implies that the whole cluster $C_{p_c,\omega}$ is contained in a strip. ∎

The corresponding statement concerning infinite star-clusters is made in the following lemma.

LEMMA 4.5. *Let $\omega \in \Omega$ and consider $I_{p_c^*}(\omega)$. Suppose $(*)$ holds in Theorem 2.2. Then each infinite open star-cluster is contained in a strip and is unbounded in both directions of the strip.*

PROOF. Consider an infinite open star-cluster C^* in $I_{p_c^*}(\omega)$. As in the proof of Theorem 3.3, this star-cluster extends to an infinite open cluster in $I_{p_c}(\omega)$. According to Lemma 4.4, this open cluster has all the properties mentioned in the lemma. Select a vertex $z \in C^*$ such that there are two paths π^1 and π^2 which tend to infinity in opposite directions of the strip starting at z and which are open in $I_{p_c}(\omega)$. The open infinite star-paths π^{1*} and π^{2*} obtained from π^1 and π^2 by decreasing p_c to p_c^* as in the proof of Theorem 3.3 also tend to infinity in opposite directions of a strip. This proves the lemma. ∎

After these lemmas, the proof of the second part of Theorem 4.1 is easy:

PROOF OF THEOREM 4.1(ii). Let $\omega \in \Omega$ and $p_c \leqq p \leqq 1 - p_c^*$. We mentioned in Section 2 that $\mu(\mathscr{P}_p) > 0$. From Theorem 3.3 it follows that also $\mu(\mathscr{P}^*_{1-p}) > 0$. Because α is irrational, there are infinitely many numbers k_i, $i \in \mathbf{N}$ such that $\omega + k_i\alpha \pmod 1 \in \mathscr{P}_p$ which means that $(k_i, 0)$ is contained in an infinite open cluster for all i. Analogously, there are infinitely many numbers l_i, $i \in \mathbf{N}$ such that $(l_i, 0)$ is contained in an infinite closed star-cluster. It follows from Lemma 4.4, Lemma 4.5 and the remark at the beginning of Section 3 that if $k_i < l_j < k_{i+1}$ for some i and j, $(k_i, 0)$ and $(k_{i+1}, 0)$ are in different open clusters. From this it follows that there are infinitely many open clusters. Note that this also shows that there are infinitely many closed star-clusters. ■

We finish this section with an application of Theorem 4.1(i). Consider the case in which $p_c = (1 + \alpha)/2$. It then follows that $p_c = 1 - p_c^*$. It follows from the proof of Theorem 4.1(i) that there are no finite open clusters in $I_p(\omega)$ for $p > p_c$. This implies that $\mathscr{P}_p = [0, p]$ for all $p > p_c$. The following lemma now suffices to show that $\mathscr{P}_{p_c} = [0, p_c]$, thereby answering a question in [M].

LEMMA 4.6. *If* $\mathscr{P}_{p+\varepsilon} = [0, p + \varepsilon]$ *for all* $1 - p > \varepsilon > 0$, *then* $\mathscr{P}_p = [0, p]$.

PROOF. Let $\varepsilon_n \downarrow 0$ as $n \to \infty$ and let $\omega \in [0, p]$. Let, for all n, π^n be such that ω percolates in $[0, p + \varepsilon_n]$ along π^n. It follows by a diagonal argument as in the proof of Lemma 4.3 that we can construct a $\pi \in \Pi$ such that ω percolates in $[0, p]$ along π. This implies that $\omega \in \mathscr{P}_p$. Since ω was arbitrary, this proves the lemma. ■

5. Finiteness of the calculation of the critical value

In this section we discuss the dynamical system M associated with the calculation of p_c in Section 2. Our aim is to prove that for almost all values of α and β, the condition (*) in Theorem 2.2 holds. To decrease the dimension, we define the following: $U' = \{(\alpha, \beta) \in \mathbf{R}^2 \mid 0 \leqq \alpha \leqq \beta \leqq \frac{1}{2}\}$ and λ' denotes the normalized Lebesgue measure on U'. Let U be the set

$$\{(x, y) \in \mathbf{R}^2 \mid 0 \leqq x \leqq y \leqq 1\}$$

and let λ denote the normalized Lebesgue measure on U. Let $f\colon U' \to U$ be defined as

$$f(\alpha, \beta) = \left(\frac{\alpha}{1-\beta}, \frac{\beta}{1-\beta}\right).$$

Then f is invertible and

$$f^{-1}(x, y) = \left(\frac{x}{1+y}, \frac{y}{1+y}\right).$$

Observe that (∗) in Theorem 2.2 is homogeneous: (∗) holds for $(\alpha_0, \beta_0, \gamma_0)$ iff, for any positive constant c, (∗) also holds for $(c\alpha_0, c\beta_0, c\gamma_0)$. Now let $\Delta \subset U$ be the set

$$\Delta = \{(x, y) \in U \mid (*) \text{ does not hold for } (x, y, 1)\}.$$

It is not difficult to see that $\Delta' = f^{-1}(\Delta) \subset U'$ is the set

$$\Delta' = \{(\alpha, \beta) \in U' \mid (*) \text{ does not hold for } (\alpha, \beta, 1-\beta)\}.$$

To prove that $\lambda'(\Delta') = 0$, it suffices to prove that $\lambda(\Delta) = 0$.

THEOREM 5.1. $\lambda(\Delta) = 0$.

The rest of this section is devoted to a proof of this theorem. Suppose $(x_0, y_0) \in U$ and consider the initial point $(x_0, y_0, 1)$. For (∗) to hold it makes no difference to divide all coefficients by the largest one after each iteration. Therefore we define $N: U \to U$ as follows. If $M(x_0, y_0, 1) = (x_0', y_0', z_0')$ then

$$N(x_0, y_0) = \left(\frac{x_0'}{z_0'}, \frac{y_0'}{z_0'}\right) =: (x_1, y_1).$$

N is given by

$$N(x, y) = \begin{cases} \left(\dfrac{x}{1-x}, \dfrac{y-x}{1-x}\right) & \text{on } \Delta_1, \\[2ex] \left(\dfrac{y-x}{1-x}, \dfrac{x}{1-x}\right) & \text{on } \Delta_2, \\[2ex] \left(\dfrac{y-x}{x}, \dfrac{1-x}{x}\right) & \text{on } \Delta_3, \end{cases}$$

where

$$\Delta_1 = \{(x, y) \in U \mid 0 \leqq 2x \leqq y\},$$

$$\Delta_2 = \{(x, y) \in U \mid x \leqq y \leqq 2x, x \leqq \tfrac{1}{2}\},$$

$$\Delta_3 = \{(x, y) \in U \mid x \geqq \tfrac{1}{2}\};$$

see Fig. 3.

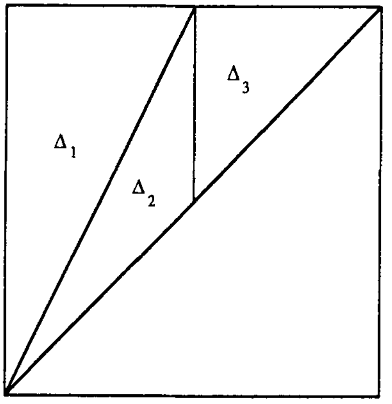

Fig. 3. The regions Δ_i, $i = 1, 2, 3$.

Let $(x_{i+1}, y_{i+1}) = N(x_i, y_i)$ for all $i \geqq 1$. Then (*) holds iff for some $i_0 \in \mathbf{N}$, $x_{i_0} + y_{i_0} \leqq 1$. Let $\Delta_- = \{(x, y) \in U \mid x + y \leqq 1\}$. We need to show that the set Δ, which can now be written as

$$\Delta = \{(x, y) \in U \mid N^n(x, y) \notin \Delta_- \text{ for all } n \geqq 0\},$$

has Lebesgue measure zero. To continue our analysis, we define the following sets:

$$\Delta_+ = U \setminus \Delta_-,$$

$$\Delta_* = \{(x, y) \in U \mid x \leqq y \leqq 3x - 1\},$$

$$B_i = \{(x, y) \in U \mid (i+1)x \leqq y \leqq (i+2)x, 1 \geqq y \geqq 1 - x\}, \quad i = 0, 1, \ldots;$$

see Fig. 4.

It is straightforward to check that N has the following properties:

(i) Δ_- is N-invariant,

(ii) N is 3 to 1,

(iii) $N(\Delta_i) = U$ for $i = 1, 2, 3$,

(iv) $N(B_i) = B_{i-1}$ for $i \geqq 1$,

(v) $N(\Delta_*) = \Delta_-$ and $N^{-1}(\Delta_-) = \Delta_- \cup \Delta_*$.

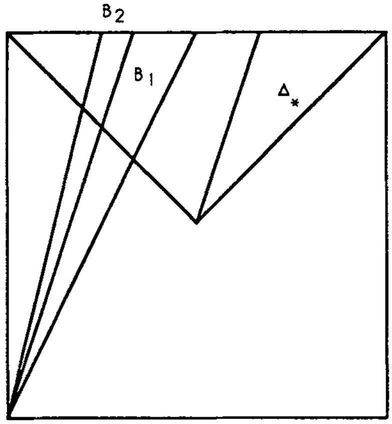

Fig. 4. The regions Δ_* and B_i, $i \in \mathrm{N}$.

To avoid the 'bad' behaviour of N in $(0, 1)$ (N does not have 'enough expansion'), we follow a rather standard procedure to define the map $T : \Delta_+ \to \Delta_+$ as follows:

$$T = \begin{cases} N^i & \text{on } B_i,\ i \geqq 1, \\ N & \text{on } \Delta_2 \cap \Delta_+ \text{ and } \Delta_3 \backslash \Delta_*, \\ T' & \text{on } \Delta_*, \end{cases}$$

where T' is an arbitrary map from Δ_* to Δ_+. It is easy to check, using the properties of N mentioned above, that

$$\Delta = \{(x, y) \in \Delta_+ \mid T^n(x, y) \notin \Delta_* \text{ for all } n \geqq 0\}.$$

We next define a 'partition' $A = \{A_i\}_{i=1}^{\infty}$ of Δ_+ as follows: $A_1 = \Delta_*^\circ$, $A_2 = (\Delta_3 \backslash \Delta_*)^\circ$, $A_3 = (\Delta_2 \cap \Delta_+)^\circ$, $A_{i+3} = B_i^\circ$, $i \geqq 1$, where C° denotes the interior of the set C. The reason why we consider interiors only is the fact that $T(\bigcup_i \delta A_i) \subset \bigcup_i \delta A_i$, where δC denotes the boundary of the set C. This means that boundaries do not play any role in this story. T is C^2 on each A_i and we denote $T \mid A_i$ by T_i. Then T_i is one-to-one and T_i^{-1} is denoted by V_i.

We need some other concepts which we define first. The Euclidean norm of a vector or matrix is denoted by $|\cdot|$, the supnorm of a matrix by $\|\cdot\|$. A map $F : E \subset \mathbf{R}^n \to \mathbf{R}^n$ is said to be *τ-expanding* if $\inf_{|u|=1} |DF(x) \cdot u| \geqq \tau > 1$ for all $x \in E$. A map $f : E \subset \mathbf{R}^n \to \mathbf{R}$ such that $|f(x)| \neq 0$ for all $x \in E$ is said to be *K-regular* if

$$\frac{|f'(x)|}{|f(x)|} < K,$$

for all $x \in E$ and for some constant K. We now have the following lemma.

LEMMA 5.2. (i) *There exists a $\tau > 1$ such that T_i is τ-expanding, uniformly in $i \geqq 2$.*

(ii) *Let JV_i denote the absolute value of the Jacobian of V_i. Then there exists a $K \in \mathbf{R}$ such that JV_i is K-regular, uniformly in $i \geqq 2$.*

PROOF. (i) We prove the assertion in the case $i = 2$, because this turns out to be the case with the least expansion. The remaining cases are treated analogously. We have

$$T_2(x, y) = \left(\frac{y - x}{x}, \frac{1 - x}{x} \right).$$

Let G denote DT_2. Then easy calculations show that

$$|G|^2 = \frac{1}{x^4}(x^2 + y^2 + 1),$$

and

$$\det G = -\frac{1}{x^3} \neq 0.$$

We will use the following formula which can be easily obtained for all invertible 2×2 matrices:

$$\| G^{-1} \|^2 = \frac{2}{|G|^2 - \sqrt{|G|^4 - 4(\det G)^2}}.$$

Let $a := |G|^2$ and $b := 2 \det G$. Then $a \geqq b$. According to the last formula it is enough to show that there exists a $\tau > 1$ such that

$$\frac{2}{a - \sqrt{a^2 - b^2}} \leqq \frac{1}{\tau},$$

uniformly in x and y (a and b depend on x and y). Rewriting this inequality yields

$$\tau \leqq \frac{a - \sqrt{a^2 - b^2}}{2}.$$

We recognize the r.h.s. to be the smallest solution of the equation

$$h(t) := t^2 - at + b^2/4 = 0.$$

It therefore suffices to show that $h(1) > 0$ which means that

$$1 - a + b^2/4 > 0.$$

To show this, we substitute the values of a and b to obtain

$$x^6 - x^2(x^2 + y^2 + 1) + 1 > 0.$$

The worst case is if $y = 1$. We then get

$$x^6 + 1 > x^2(2 + x^2).$$

This should be true for all $x \in [\frac{1}{2}, \frac{2}{3}]$. It is easy to check that $x = \frac{2}{3}$ is the worst case, in which we obtain

$$1 + \frac{64}{729} > 1 + \frac{63}{729}.$$

It is a narrow escape but it works and it proves the assertion for $i = 2$. The other cases are treated analogously. For $i > 3$, one obtains a uniform τ, independent of i.

(ii) For a change, we prove this assertion in the case $i \geqq 4$. Straightforward calculations show that

$$V_i(\xi, \eta) = \left(\frac{\xi}{1 + i\xi}, \frac{\eta + i\xi}{1 + i\xi}\right).$$

Then $JV_i = 1/(1 + i\xi)^3$ and $JV_i' = (-3i/(1 + i\xi)^4, 0)$, so

$$\left|\frac{JV_i'}{JV_i}\right| = \frac{3i}{1 + i\xi} \leqq \frac{3i}{1 + \frac{1}{2}i} \leqq 6,$$

uniformly in i.

The remaining cases are treated analogously. ∎

The following lemma might be deduced from Lemma 5.2, but also follows from straightforward calculations.

Lemma 5.3. *$|JV_i'(\xi)/JV_i(\eta)| < K'$, uniformly in $i \geqq 2$ and for all $\xi, \eta \in TA_i$, where K' is some constant.*

Let us now explain how to use these notions to prove Theorem 5.1. Write, for all $k \geqq 0$,

$$\Delta(k) = \{(x, y) \in \Delta_+ \mid T^i(x, y) \notin \Delta_* \text{ for all } 0 \leqq i \leqq k\}.$$

Then $\Delta(k) \downarrow \Delta$ as $k \to \infty$. It therefore suffices to show that $\lambda(\Delta(k))$ goes down exponentially fast. For this we introduce the following notation.

A cylinder set $[c^k] = [c_0, \ldots, c_k]$ is defined as

$$[c^k] = \{(x, y) \in \Delta_+ \mid T^i(x, y) \in A_{c_i} \text{ for } 0 \leqq i \leqq k\}.$$

If $(x, y) \in \Delta(k)$, there is a unique cylinder set $[c^k]$ such that $(x, y) \in [c^k]$ and $c_i \neq 1$, $i = 0, \ldots, k$. To prove that $\lambda(\Delta(k))$ goes down exponentially fast, we will show that

$$(**) \qquad \frac{\lambda[c_0, c_1, \ldots, c_k, 1]}{\lambda[c_0, \ldots, c_k]} \geqq L' > 0,$$

for all cylinder sets $[c^k]$ such that $c_i \neq 1$, $i = 0, \ldots, k$ and where L' is a constant independent of k and c^k. To prove (**), observe that $T^{k+1}[c^k]$ is either Δ_+ or $A_1 \cup A_2 \cup A_3$. The former occurs iff $c_k \in \{2, 3\}$. Writing V_c^k for the inverse of T^k on $T^k[c^k]$, we have

$$\frac{\lambda[c_0, \ldots, c_k, 1]}{\lambda[c_0, \ldots, c_k]} = \frac{\int_{A_1} JV_c^k(\xi) d\lambda(\xi)}{\int_E JV_c^k(\eta) d\lambda(\eta)},$$

where E denotes either Δ_+ or $A_1 \cup A_2 \cup A_3$. To prove (**) and therefore Theorem 5.1, it now suffices to show that the quotients of the integrands in this formula uniformly stay away from zero and infinity. The following lemma deals with this property and is sufficient to prove Theorem 5.1.

Lemma 5.4. *For all $k \in \mathbf{N}$, $[c^k]$, and $\xi, \eta \in T^k[c^k]$, we have*

$$\frac{1}{L''} \leqq \frac{JV_c^k(\xi)}{JV_c^k(\eta)} \leqq L'',$$

where L'' does not depend on ξ, η, k or $[c^k]$.

Proof. We write V for a suitable V_i and V^r for a suitable composition $V_{l_r} \circ \cdots \circ V_{l_1}$. Then we have

$$\frac{JV_y^k(\xi)}{JV_y^k(\eta)} = \prod_{r=1}^{k-1} \frac{JV(V^r\xi)}{JV(V^r\eta)} = \prod_{r=1}^{k-1} \left(\frac{JV(V^r\xi) - JV(V^r\eta)}{JV(V^r\eta)} + 1 \right).$$

We have

$$|JV(V^r\xi) - JV(V^r\eta)| \leqq |JV'(\phi)| \cdot |V^r\eta - V^r\xi| \leqq |JV'(\phi)| \cdot \frac{|\eta - \xi|}{\tau^n},$$

by the τ-expansiveness of T, where ϕ denotes a suitable point.

By uniform regularity of JV_i and Lemma 5.3, we have

$$\frac{JV_y^k(\xi)}{JV_y^k(\eta)} \leqq \prod_{r=1}^{k-1}\left(\frac{|JV'(\phi)|}{|JV(V^r\eta)| \cdot \tau^r} + 1\right)$$

$$\leqq \exp \sum_{r=1}^{k-1} \log\left(\frac{K'}{\tau^r} + 1\right) \leqq \exp\left(\frac{\tau K'}{\tau - 1}\right),$$

which proves the lemma. ■

The reason why we switched from N to T is the fact that N is not uniformly expanding. This is easy to see, as $DN(0, 1) = \mathrm{id}$. The proof above does not work if we replace T by N.

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References

[BK] J. van den Berg and M. S. Keane, *On the continuity of the percolation probability function*, in *Conference on Modern Analysis and Probability* (R. Beals et al., eds.), Contemporary Mathematics 26, American Mathematical Society, Providence, RI, 1984, pp. 61–65.

[BKa] R. M. Burton and M. Keane, *Density and uniqueness in percolation*, Commun. Math. Phys. **121** (1989), 501–505.

[BKb] R. M. Burton and M. Keane, *Topological and metric properties of infinite clusters in stationary two-dimensional site percolation*, preprint (1988).

[H] J. M. Hammersley, *A Monte Carlo solution of percolation in the cubic crystal*, in *Methods in Computational Physics*, Vol. 1 (B. Adler et al., eds.), Acadamic Press, New York, 1963.

[K] H. Kesten, *Percolation Theory for Mathematicians*, Birkhauser, Boston, 1982.

[M] R. W. J. Meester, *An algorithm for calculating critical probabilities and percolation functions in percolation models defined by rotations*, Ergodic Theory and Dynamical Systems, to appear.

[NS] C. M. Newman and L. S. Schulman, *Infinite clusters in percolation models*, J. Stat. Phys. **26** (1981), 613–628.

[R] L. Russo, *On the critical percolation probabilities*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **56** (1981), 229–237.

[SE] M. F. Sykes and J. W. Essam, *Exact critical percolation probabilities for site and bond problems in two dimensions*, J. Math. Phys. **5** (1964), 1117–1127.